CLAREMONT REU ABSTRACT OF L. FUKSHANSKY'S GROUP: ON WELL-ROUNDED SUBLATTICES OF THE HEXAGONAL LATTICE

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Kepler's Conjecture, recently proved by T. Hales, states that the densest packing of spheres in 3-space has spheres centered along the face-centered cubic (fcc) lattice. A lattice is a free $\mathbb{Z}$-module formed by taking the span of a collection of linearly independent vectors in $\mathbb{R}^N$ over the integers. The two-dimensional analogue of Kepler's Conjecture, proved by L. F. Toth in 1940, states that the densest packing of circles in the plane is the hexagonal lattice (see [2], [7] for an overview of the area, and [3] for introductory lecture notes). We may define the hexagonal lattice as the set of all integral linear combinations of the vectors

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/2 \\ \sqrt{3}/2 \end{bmatrix}.$$

This lattice solves a variety of discrete optimization problems (see [4] for some details), and naturally quite some work has been done by different authors investigating the structure of the hexagonal lattice. Notably, Bernstein, Sloane, and Wright [1] studied the distribution of sublattices (subsets of lattices which are lattices themselves) of this lattice in an attempt to solve certain problems related to digital communications. Their paper emphasizes a particular class of sublattices, called ideal sublattices: these sublattices are defined as the images of ideals in the ring of Eisenstein Integers under the canonical mapping $\mathbb{C} \mapsto \mathbb{R}^2$. During this REU program, we attempted to extend their work on ideal sublattices to a more general class of well-rounded (a lattice in $\mathbb{R}^N$ is called well-rounded if its set of vectors with minimal nonzero norm spans $\mathbb{R}^N$) sublattices of the hexagonal lattice (well-rounded sublattices of $\mathbb{Z}^2$ have previously been studied in [5], [6]).

To extend the work of Bernstein, Sloane, and Wright, Daniel Moore primarily focused on extending the results maximizing signal-to-noise ratio (SNR). SNR of a lattice is directly related to its Epstein zeta function (a classical Dirichlet series generating function of the square of the norm on a lattice) evaluated at 2. Since every lattice is an additive group, we can define the index of a sublattice to be the group index. Moore was therefore addressing the following two questions, providing partial answers to both of them:

1. Given the set of well-rounded sublattices of fixed index, which one of them has the largest minimal nonzero norm?
2. Given the set of well-rounded sublattices of fixed index, which one of them minimizes the value of Epstein zeta at 2?

To answer these questions, Moore needed to understand what the well-rounded sublattices of the hexagonal lattice looked like.

The problem of understanding the structure of the set of well-rounded sublattices of the hexagonal lattice was addressed by the other two members of the group. Andrew Ohana attacked this problem by attempting to parameterize the well-rounded sublattices. He related the similarity classes of well-rounded sublattices (two lattices are called similar if they can be obtained from each other by rotation and dilation; this is an equivalence relation on the set of lattices, which preserves well-roundedness) to the rational points on a certain ellipse, and thus was able to parameterize them. With this and the ability to find lattices with minimal index in each similarity class, Ohana gave a foundation for further work.

Ohana's parameterization was, however, somewhat abstract. To develop a geometric understanding of the distribution of these sublattices, Whitney Zeldow took a more combinatorial approach. In prior work by L. Fukshansky [6], a combinatorial parameterization of similarity classes of well-rounded sublattices of the integer lattice was given with the use of certain matrices acting on the space of primitive Pythagorean triples. Suspecting that an analogous structure exists for the well-rounded sublattices of the hexagonal lattice, Zeldow committed a large amount of time trying to find appropriate matrices. She managed to find a set which appeared to satisfy the desired properties and made some interesting conjectures. With her work, the group gained some intuition about the growth rate of the index of a sublattice inside of a fixed similarity class, and outlined some questions for future investigation.

References

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